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Role of strong electric fields in the generation of harmonics

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Abstract. The nonlinear evaluation of the reflected and the transmitted fields of the fundamental and third harmonic waves, which are generated by the interaction of an intense microwave with a weakly ionized nitrogen plasma slab, has been carried out by solving the nonlinear Maxwell equations numerically using a Runge–Kutta method. We find that an increase in the fundamental field increases the harmonic output. For an incident field of about 3 esu and of frequency $\omega = 10^{10}\,\mathrm{rad\,s^{-1}}$, the peak power of the third harmonic obtained is about 0·04 % of the fundamental which is in fairly good agreement with experiment. The harmonic output also increases with electron concentration in the range $\omega < \omega_p < 3\omega$ (where ω_p is the plasma frequency). The harmonic fields show resonant increase at fundamental and at harmonic frequencies.

1. Introduction

The nonlinear propagation of high frequency electromagnetic signals in weakly ionized plasmas has attracted the attention of many workers (Krenz 1965, Tang 1966, Vogel and Desloge 1968, Osagarvara and Mori 1969, Sodha and Kaw 1969) in recent years owing to its application in multiple frequency generators, plasma diagnostics etc. Most of these investigations have been limited to moderately strong fields which do not affect the plasma parameters appreciably. The solution of the Maxwell equations for such a nonlinear medium is obtained on the basis of the perturbation approximation in which the change in free-carrier parameters is assumed to be small.

Jayaram and Tripathi (1971) have recently investigated the nonlinear response of a magnetoplasma to intense electromagnetic waves beyond the perturbation limit. The fundamental and harmonic components of current density were evaluated. These calculations of harmonic current density are, however, insufficient to give a correct estimate of the generated electric field because the propagation parameters of the fundamental and harmonic waves are strongly field dependent and may even change the qualitative behaviour of the generated field. Nevertheless, the evaluation of the electric field presents a serious problem of solving the nonlinear Maxwell equations.

Papa and Haskell (1966) have used a Runge–Kutta method (Kuo 1965) for the numerical evaluation of the fundamental fields in an inhomogeneous magnetoplasma slab on which an elliptically polarized electromagnetic wave is incident normally. The DC magnetic field and electron density gradient are perpendicular to the interfaces. They have obtained the reflection and transmission coefficients of the fundamental wave for the two cases of constant mean free path and constant collision frequency.

By using a Runge–Kutta method, we have numerically evaluated the harmonic fields in a magnetoplasma when an intense microwave interacts with it. In the following section expressions for the fundamental and third harmonic current densities have been obtained. In § 3 a numerical evaluation of the fundamental and third harmonic fields, for a nitrogen plasma slab at 300 K, has been carried out. A discussion of the results obtained and their comparison with experiment follows in § 4.

2. Fundamental and third harmonic current densities

Let us consider a slab of homogeneous weakly ionized isotropic plasma, dominated by electron–neutral particle collisions. A microwave is incident normally on its face at $z = 0$; the z axis being the direction of propagation of the wave. The electric vector of the wave is in the direction of the x axis. The other boundary of the plasma is at $z = d$. A static magnetic field $\boldsymbol{H}_0$ is applied along the z direction.

As a result of microwave interaction with the plasma, the velocity distribution of electrons is changed and is given by (cf equation (8a) of Jayaram and Tripathi 1971)

$$f_0^0 = N_0 \exp\left(- \int_0^u \frac{2u\,\mathrm{d}u}{1+\alpha(u)} \right), \tag{1}$$

where u is the dimensionless electron velocity; α, the nonlinearity parameter, is given by

$$\alpha(u) = \frac{e^2 M}{12 m^2 kT} \left(\frac{A_1^1 A_1^{1*}}{\nu^2 + (\omega - \omega_e)^2} + \frac{A_2^1 A_2^{1*}}{\nu^2 + (\omega + \omega_e)^2} \right), \tag{2}$$

and $A_{1,2}^1$ are fields of extraordinary and ordinary modes of the fundamental wave given by

$$A_{1,2}^1 = E_x^1 \pm \mathrm{i} E_y^1, \tag{3}$$

the other symbols have their usual meaning.

On using expression (1) for the distribution function we obtain the following expressions for the fundamental J^1 and third harmonic J^{111} current density (cf equation (12) and (14) respectively of Jayaram and Tripathi 1971):

$$J_x^1 \pm \mathrm{i} J_y^1 = \sigma_{m\pm}^1 A_{1,2}^1 \tag{4}$$

and

$$J_x^{111} \pm \mathrm{i} J_y^{111} = \sigma_{m\pm}^{111} A_{1,2}^{111} + \gamma_\pm (A_{1,2}^1)^2 (A_{2,1}^1), \tag{5}$$

where $\sigma_{m\pm}$ are the conductivity components for the two modes. For the fundamental wave

$$\sigma_{m\pm}^1 = \tfrac{2}{3}\omega_p^2 \left(\frac{2kT}{m} \right)^{3/2} \int_0^\infty \frac{u^4 f_0^0}{1+\alpha(u)} \left(\frac{1}{\nu + \mathrm{i}(\omega \mp \omega_e)} \right) \mathrm{d}u \tag{6}$$

and for the third harmonic wave

$$\sigma_{m\pm}^{111} = \tfrac{2}{3}\omega_p^2 \left(\frac{2kT}{m} \right)^{3/2} \int_0^\infty \frac{u^4 f_0^0}{1+\alpha(u)} \left(\frac{1}{\nu + \mathrm{i}(3\omega \mp \omega_e)} \right) \mathrm{d}u. \tag{7}$$

The coupling coefficients $\gamma_\pm$ of equation (5) are given by

$$\gamma_\pm = \frac{\omega_p^2 e}{576 m} (I_1 + I_2 + I_3), \tag{8}$$

where I_1, I_2 and I_3 are integrals given by

$$I_1 = 8\pi\left(\frac{2kT}{m}\right)^{1/2}\frac{1}{i\omega}\int_0^\infty \frac{\nu+i\omega}{(\nu+i\omega_{1\mp})(\nu+i\omega_{1\pm})}\frac{\partial}{\partial u}u^2\frac{\partial}{\partial u}\frac{1}{u^2}\frac{\partial}{\partial u}\left(\frac{u^3}{\nu+i\omega_{3\mp}}\right)f_0^0\,\mathrm{d}u, \tag{9}$$

$$I_2 = \frac{16\pi}{5}\left(\frac{2kT}{m}\right)^{1/2}\int_0^\infty \frac{\nu+i\omega}{(\nu+i\omega_{1\mp})(\nu+i\omega_{1\pm})}\frac{\partial}{\partial u}\frac{1}{u}\frac{\partial}{\partial u}\left(\frac{u^4}{\nu+2i\omega}\frac{\partial}{\partial u}\frac{1}{\nu+i\omega_{2\mp}}\right)f_0^0\,\mathrm{d}u \tag{10}$$

and

$$I_3 = \frac{48\pi}{5}\left(\frac{2kT}{m}\right)^{1/2}\int_0^\infty \frac{1}{\nu+i\omega_{1\mp}}\frac{\partial}{\partial u}\frac{1}{u}\frac{\partial}{\partial u}\left(\frac{u^4}{\nu+2i\omega_{1\mp}}\frac{\partial}{\partial u}\frac{1}{\nu+i\omega_{3\mp}}\right)f_0^0\,\mathrm{d}u, \tag{11}$$

with

$$\omega_{n\pm} = n\omega \pm \omega_e. \tag{12}$$

The integrals appearing in equations (7) and (8) can be numerically evaluated for various velocity dependences of the collision frequency ν.

3. Evaluation of electric fields by the Runge–Kutta method

The necessary Maxwell equations governing the electric and the magnetic vectors are

$$\nabla\times\boldsymbol{E} = -\frac{\mu}{c}\frac{\partial\boldsymbol{H}}{\partial t} \tag{13}$$

and

$$\nabla\times\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J}+\frac{1}{c}\frac{\partial\boldsymbol{D}}{\partial t}, \tag{14}$$

where μ is the magnetic permeability of the medium ($\mu \simeq 1$ for plasmas) and $\boldsymbol{D}$ is the electric displacement vector. For fundamental fields these equations, on separating into real and imaginary components, can be written as

$$\frac{\partial A^1_{1,2\mathrm{r}}}{\partial\tau} = \mp H^1_{1,2\mathrm{i}}, \tag{15}$$

$$\frac{\partial A^1_{1,2\mathrm{i}}}{\partial\tau} = \mp H^1_{1,2\mathrm{r}}, \tag{16}$$

$$\frac{\partial H^1_{1,2\mathrm{r}}}{\partial\tau} = \pm S_{1,2\mathrm{r}}A^1_{1,2\mathrm{r}} \mp S_{1,2\mathrm{i}}A^1_{1,2\mathrm{i}} \tag{17}$$

and

$$\frac{\partial H^1_{1,2\mathrm{i}}}{\partial\tau} = \pm S_{1,2\mathrm{i}}A^1_{1,2\mathrm{r}} \pm S_{1,2\mathrm{r}}A^1_{1,2\mathrm{i}}, \tag{18}$$

where

$$H^1_{1,2} = H^1_x \pm iH^1_y, \tag{19}$$

$$S_{1\mathrm{r}}+iS_{1\mathrm{i}} = 1-\frac{4\pi i}{\omega}\sigma^1_{m+}, \tag{20}$$

$$S_{2r} + iS_{2i} = 1 - \frac{4\pi i}{\omega}\sigma^{1}_{m-} \tag{21}$$

and $\tau = z\omega/c$. The subscripts r and i refer to the real and imaginary components of the quantities.

For the third harmonic components we can write equations (13) and (14) as

$$\frac{\partial A^{111}_{1,2r}}{\partial \tau} = \mp 3H^{111}_{1,2r}, \tag{22}$$

$$\frac{\partial A^{111}_{1,2i}}{\partial \tau} = \mp 3H^{111}_{1,2i}, \tag{23}$$

$$\frac{\partial H^{111}_{1,2r}}{\partial \tau} = \pm S_{3,5r}A^{111}_{1,2r} \mp S_{3,5i}A^{111}_{1,2i} \mp S_{4,6r}[\{(A^{1}_{1,2r})^2 - (A^{1}_{1,2i})^2\}A^{1}_{2,1r} - 2A^{1}_{1,2r}A^{1}_{1,2i}A^{1}_{2,1i}]$$
$$\pm S_{4,6i}[\{(A^{1}_{1,2r})^2 - (A^{1}_{1,2i})^2\}A^{1}_{2,1i} + 2A^{1}_{1,2r}A^{1}_{1,2i}A^{1}_{2,1r}] \tag{24}$$

and

$$\frac{\partial H^{111}_{1,2i}}{\partial \tau} = \pm S_{3,5i}A^{111}_{1,2r} \pm S_{3,5r}A^{111}_{1,2i} \mp S_{4,6i}[\{(A^{1}_{1,2r})^2 - (A^{1}_{1,2i})^2\}A^{1}_{2,1r} - 2A^{1}_{1,2r}A^{1}_{1,2i}A^{1}_{2,1i}]$$
$$\mp S_{4,6r}[\{(A^{1}_{1,2r})^2 - (A^{1}_{1,2i})^2\}A^{1}_{2,1i} + 2A^{1}_{1,2r}A^{1}_{1,2i}A^{1}_{2,1r}], \tag{25}$$

where

$$S_{3r} + iS_{3i} = 3 - \frac{4\pi i}{\omega}\sigma^{111}_{m+}, \qquad S_{4r} + iS_{4i} = \frac{4\pi i}{\omega}\gamma_{+}, \tag{26}$$

$$S_{5r} + iS_{5i} = 3 - \frac{4\pi i}{\omega}\sigma^{111}_{m-}, \qquad S_{6r} + iS_{6i} = \frac{4\pi i}{\omega}\gamma_{-}. \tag{27}$$

We may point out that in the limit of very weak collision frequency and vanishing magnetic fields, equation (24) reduces to

$$\frac{\partial^2 A^{111}_{1r}}{\partial \tau^2} - 3S_{3r}A^{111}_{1r} = 0, \tag{28}$$

which has a solution of the form,

$$A^{111}_{1r} = C\exp\{-(3S_{3r})^{1/2}\tau\}, \qquad \text{for } S_{3r} < 0 \tag{29}$$

where C is some constant. As is clearly seen from equation (26), this solution is valid only if $(4\pi i\sigma^{111}_{m+}/\omega) > 3$. As σ^{111}_{m+} decreases with decrease in ω_p, below a certain value of ω_p (say ω_p^*) this inequality is no longer satisfied. For this reason the curves corresponding to very weak collisions for low density cases in figures 1 to 5 have not been given.

Using equations (15) to (27) it is possible to compute the electromagnetic fields of the fundamental and third harmonic waves inside and outside the plasma slab. This is accomplished by assuming a value for the fundamental and third harmonic components of the transmitted fields at $z = d$ (because only forward propagating waves exist for $z > d$) and performing a fourth order Runge–Kutta step-by-step integration of Maxwell's equations backwards through the plasma slab. By matching the tangential field components across the two faces, $z = 0$ and $z = d$, of the slab it is possible to construct expressions for the reflected and transmitted fundamental and third harmonic

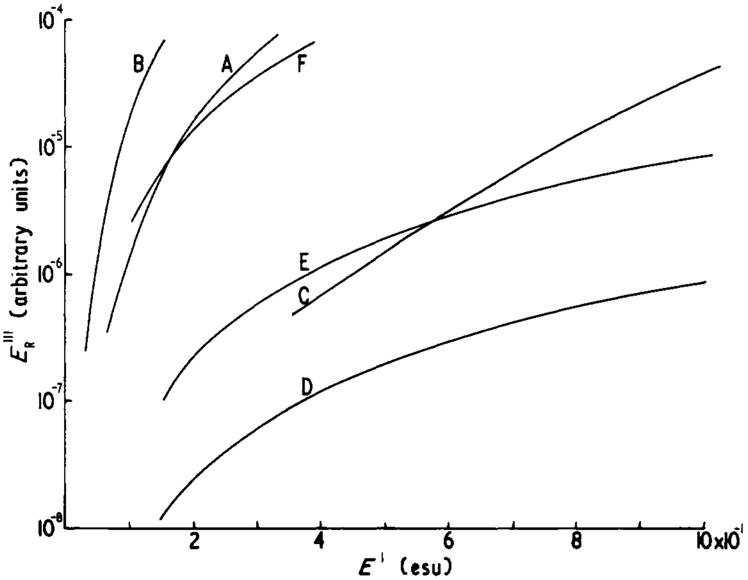

Figure 1. Variation of reflected third harmonic electric field with incident field in the absence of an external magnetic field. Curves A, B and C correspond to $(\omega_p/\omega)^2 = 0{\cdot}1$, 1 and 10 respectively and $(\nu_0/\omega)^2 = 0{\cdot}9$ while curves D, E and F correspond to the same $(\omega_p/\omega)^2$ and $(\nu_0/\omega)^2 = 10$. For curve C the ordinate is to be multiplied by 10^3.

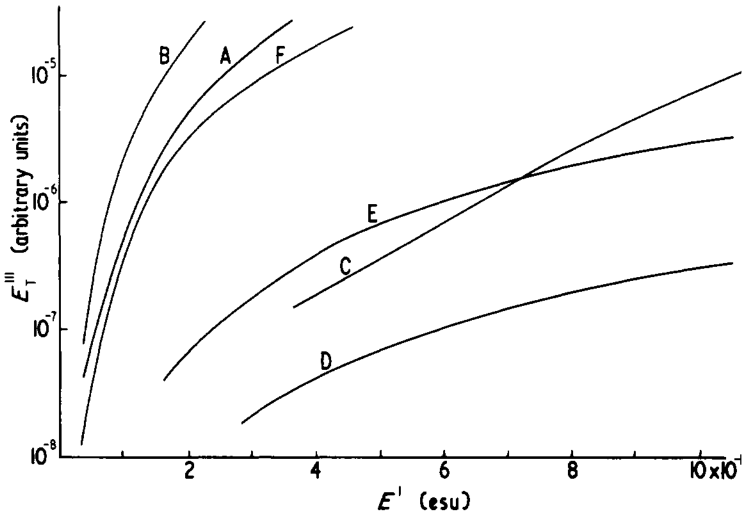

Figure 2. Variation of transmitted third harmonic electric field with incident field in the absence of an external magnetic field. Curves A, B and C correspond to $(\omega_p/\omega)^2 = 0{\cdot}1$, 1 and 10 respectively and $(\nu_0/\omega)^2 = 0{\cdot}9$ while curves D, E and F correspond to the same $(\omega_p/\omega)^2$ and $(\nu_0/\omega)^2 = 10$. For curve C the ordinate is to be multiplied by 10^3.

waves. The reflection and transmission coefficients for the fundamental components of the fields are given by

$$R^{1}_{\mathrm{E,O}} = 4\frac{A^{1}_{1,2} \pm \mathrm{i}H^{1}_{1,2}}{A^{1}_{1,2} \mp \mathrm{i}H^{1}_{1,2}} \tag{30}$$

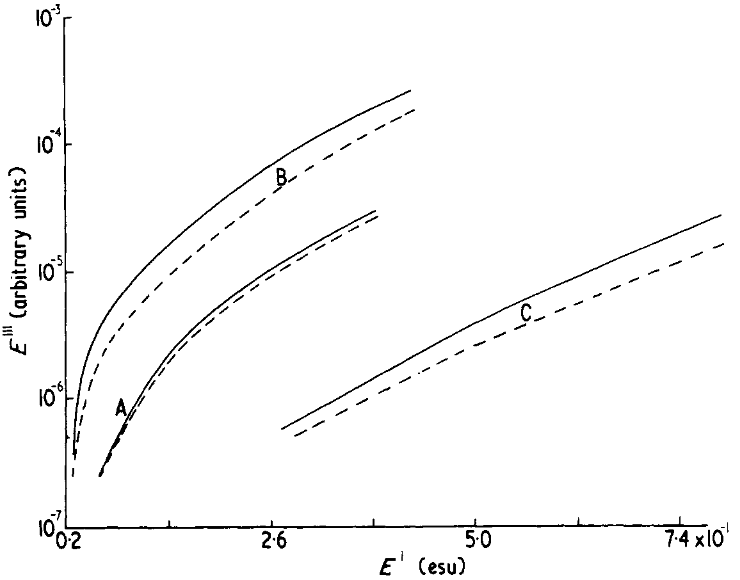

Figure 3. Variation of reflected (full curves) and transmitted (broken curves) third harmonic electric field (extraordinary component) with incident field. Curves A, B and C correspond to $(\omega_p/\omega)^2 = 0{\cdot}1$, 1 and 10 respectively, $(v_0/\omega)^2 = 0{\cdot}9$ and $(\omega_e/\omega) = 0{\cdot}9$. For curve C the ordinate is to be multiplied by 10^2.

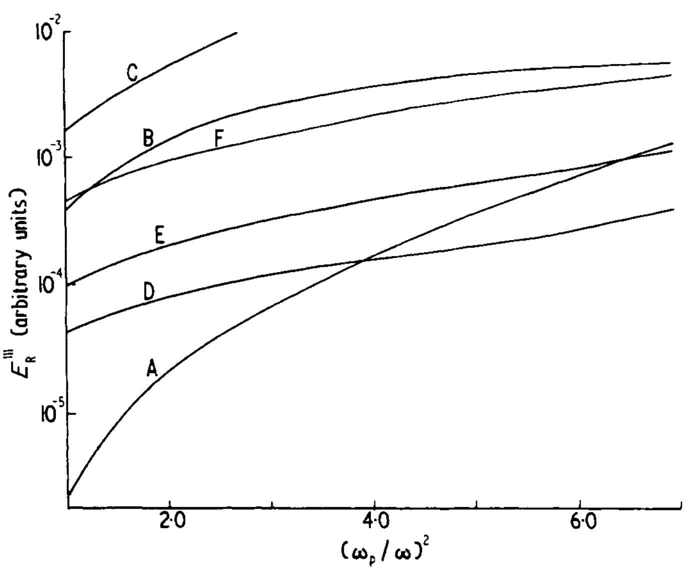

Figure 4. Variation of reflected third harmonic field (extraordinary component) with normalized plasma frequency for incident field 2·2 esu. Curves A, B and C correspond to $(\omega_e/\omega) = 0{\cdot}5$, 1 and 3 respectively and $(v_0/\omega)^2 = 0{\cdot}9$, while curves D, E and F correspond to the same (ω_e/ω) and $(v_0/\omega)^2 = 10$.

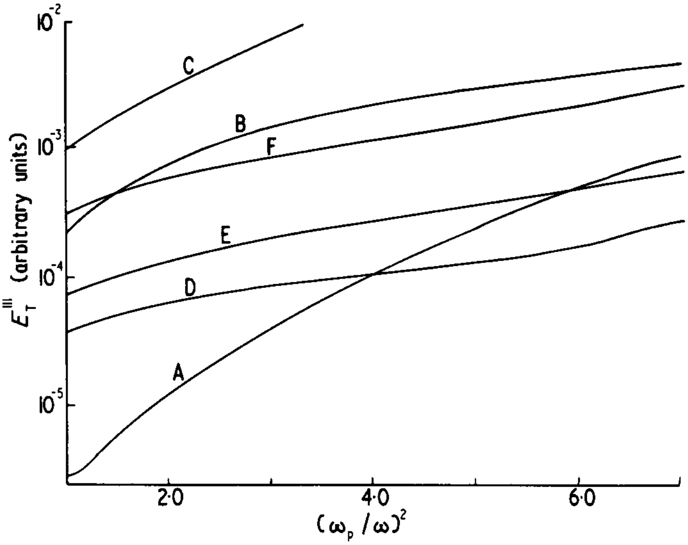

Figure 5. Variation of transmitted third harmonic field (extraordinary component) with normalized plasma frequency for incident field 2·2 esu. Curves A, B and C correspond to $(\omega_e/\omega) = 0{\cdot}5$, 1 and 3 respectively and $(v_0/\omega)^2 = 0{\cdot}9$, while curves D, E and F correspond to the same (ω_e/ω) and $(v_0/\omega)^2 = 10$.

and

$$T^1_{\mathrm{E,O}} = \frac{(A^1_{1,2})_d}{A^1_{1,2} \mp \mathrm{i}H^1_{1,2}}, \tag{31}$$

where the subscripts O and E refer to the ordinary and extraordinary waves and the fields appearing on the right-hand side of equations (30) and (31) correspond to $z = 0$ except $(A^1_{1,2})_d$ which corresponds to $z = d$.

The evaluation of third harmonic fields is somewhat complex because the third harmonic field assumed at $z = d$ must be such that the incident field for $z < 0$ vanishes. To fulfil this condition we adopt a reverse procedure. Assuming some arbitrary value of the third harmonic extraordinary or ordinary mode at $z = d$, by using the Runge–Kutta method we evaluate the incident fields at $z = 0$ in two ways: (*a*) by neglecting the nonlinear source term, that is, taking $\gamma_\pm = 0$; and (*b*) taking $\gamma_\pm$ as permitted by the fundamental fields. The former calculation gives the reflection and transmission coefficients for the third harmonic when nonlinear propagation characteristics introduced by the fundamental wave have been included but the source term taken to be zero. When $\gamma_\pm$ is taken as given by equation (8) we obtain the corresponding reflected and transmitted fields. We thus obtain the following expressions for the third harmonic reflected and transmitted fields:

$$E^{111}_{\mathrm{R}1,2} = \frac{A^{111}_{1,2} \pm \mathrm{i}H^{111}_{1,2}}{2} - \frac{1}{2}\,\frac{BA^{111}_{1,2} \pm \mathrm{i}BH^{111}_{1,2}}{BA^{111}_{1,2} \mp \mathrm{i}BH^{111}_{1,2}}(A^{111}_{1,2} \mp \mathrm{i}H^{111}_{1,2}) \tag{32}$$

and

$$E^{111}_{\mathrm{T}1,2} = (A^{111}_{1,2})_d\left(1 - \frac{A^{111}_{1,2} \mp \mathrm{i}H^{111}_{1,2}}{BA^{111}_{1,2} \mp \mathrm{i}BH^{111}_{1,2}}\right), \tag{33}$$

where $BA_{1,2}^{111}$ and $BH_{1,2}^{111}$ are the third harmonic electric and magnetic fields of extraordinary and ordinary modes at $z = 0$ when $\gamma_{\pm} = 0$ and $(A_{1,2}^{111})_d$ corresponds to the generated fields at $z = d$.

Numerical evaluation of the third harmonic fields has been carried out for a nitrogen plasma slab of thickness 3 cm at 300 K on an IBM-360 computer. We chose a nitrogen plasma to facilitate comparison of our results with experimental results available. The velocity dependence of the collision cross section for nitrogen has been taken into account, by taking $\nu = \nu_0 u^2$.

4. Discussion

A rigorous physical interpretation of the results is very difficult because of the overlap of a large number of competing processes. Nevertheless the effect of various plasma parameters may be understood in the following manner.

From figures 1 and 2 we see that as the incident field increases the harmonic output also increases. The effect of increasing the field on the collision frequency, through the electron temperature, is to increase the collisions. Normally the effect of increasing the collisions is to decrease the generated fields by lowering the power absorption. This means that the effects appearing through the propagation parameters are more important

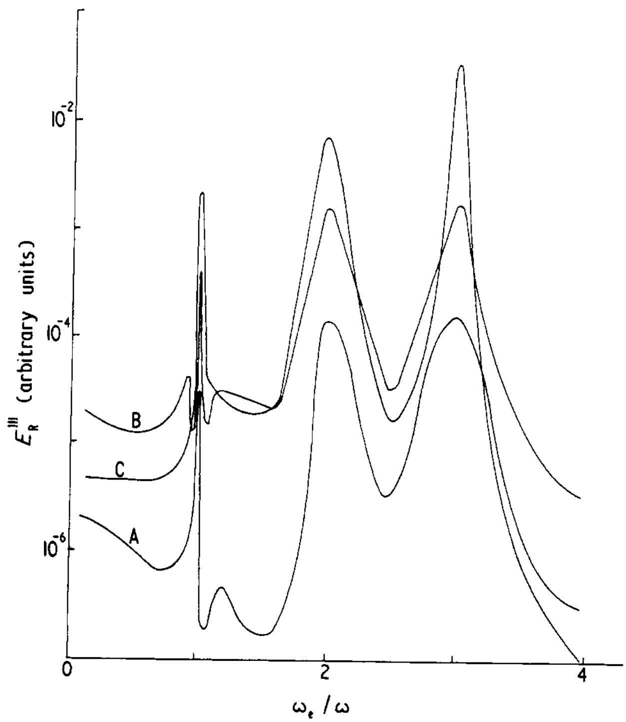

Figure 6. Variation of reflected third harmonic field (extraordinary component) with normalized cyclotron frequency for an incident field of 2·2 esu. Curves A and B correspond to $(\omega_p/\omega)^2 = 0{\cdot}1$ and 1 respectively and $(\nu_0/\omega)^2 = 0{\cdot}9$ while curve C correspond to $(\omega_p/\omega)^2 = 1$ and $(\nu_0/\omega)^2 = 0{\cdot}02$.

than the collision frequency dependence on the field. In the presence of a magnetic field (figure 3) the harmonic output shows a similar behaviour with fundamental input.

An increase in the electronic concentration is known to increase the nonlinear source term linearly for harmonic generation (Ginzburg 1960). This can also be seen from figures 4 and 5 where the harmonic output increases with ω_p for the range $\omega < \omega_p < 3\omega$. The harmonic output at weak collisions ($\nu \sim \omega$) is more than that at strong collisions ($\nu > \omega$).

The effect of a static magnetic field shown in figures 6 and 7 is obviously to enhance the fundamental and harmonic current densities at various cyclotron resonances. At cyclotron resonances the harmonic power of the extraordinary mode has been found to increase while at off-resonance the effects are less significant. Cyclotron resonance effects (at $\omega = \omega_e$ and $2\omega = \omega_e$) have also been found to appear in the ordinary mode of the harmonic owing to the dependence of this mode on both modes of the fundamental wave.

It may be mentioned here that we have not discussed the size effects (dimension resonances) as they are expected to be similar to the ones discussed by Jayaram and Tripathi (1970).

The experimental results for third harmonic fields are rarely available, therefore a direct comparison of our results is not possible. The measurements of Inada *et al* (1960) and the calculations of Varnum and Desloge (1969) (based on estimates of harmonic current density, involving the inelastic collisions also) do not give any assessment of the reflected and transmitted fields. Nevertheless, the ratio of peak power P_3/P_1 obtained by Inada *et al* (1960) and Varnum and Desloge (1969) for $\omega = 1{\cdot}88 \times 10^{10}$ rad s^{-1} and

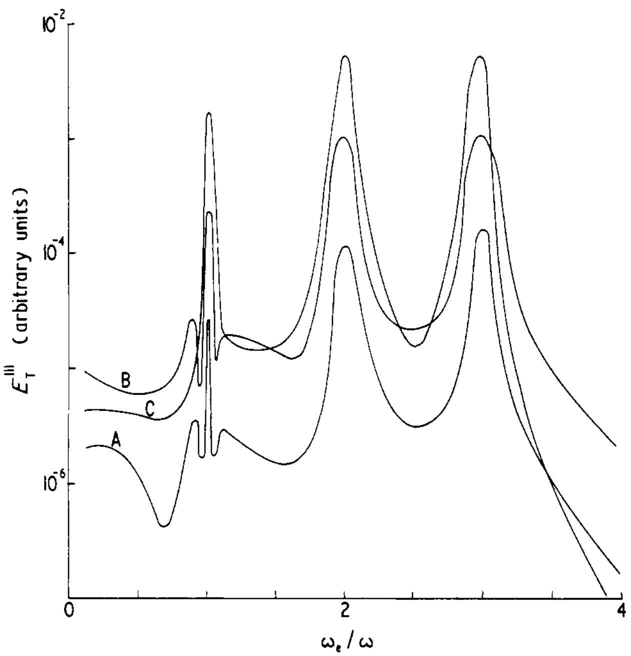

Figure 7. Variation of transmitted third harmonic field (extraordinary component) with normalized cyclotron frequency for an incident field of 2·2 esu. Curves A and B correspond to $(\omega_p/\omega)^2 = 0{\cdot}1$ and 1 respectively and $(\nu_0/\omega)^2 = 0{\cdot}9$ while curve C corresponds to $(\omega_p/\omega)^2 = 1$ and $(\nu_0/\omega)^2 = 0{\cdot}02$.

$E = 10^5\ \mathrm{V\,m^{-1}}$ are $1{\cdot}2 \times 10^{-3}$ and $2{\cdot}2 \times 10^{-3}$ respectively, whereas in our case this turns out to be $0{\cdot}4 \times 10^{-3}$. We see that this is in good agreement with the experimental results. The small discrepancy between our results and those of Inada *et al* (1960) can be attributed to the neglect of inelastic collisions in our analysis which may be important at high fields.

Acknowledgments

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